ONE-DIMENSIONAL DEGENERATE DIFFUSION OPERATORS

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Abstract. The aim of this paper is to present some results about generation, sectoriality and gradient estimates both for the semigroup and for the resolvent of suitable realizations of the operators

$$A^{\gamma,b}u(x) = \gamma x u''(x) + bu'(x),$$

with constants $\gamma > 0$ and $b \geq 0$, in the space $C([0,\infty])$.

The motivation for this paper comes from investigations on the analiticity in the space of continuous functions on the d-dimensional canonical simplex S^d of the semigroup generated by the multi-dimensional Fleming-Viot operator (also known as Kimura operator or Wright-Fischer operator, see [15, 16, 17, 20, 24, 25])

$$Au(x) = \frac{1}{2}\gamma(x)\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j}u(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}u(x), \tag{0.1}$$

where $b = (b_1, \ldots, b_d)$ is a continuous inward pointing drift and γ a strictly positive continuous function on S^d. The operator (0.1) arises in the theory of Fleming–Viot processes as the generator of a Markov C_0–semigroup defined on $C(S_d)$. Fleming–Viot processes are measure–valued processes that can be viewed as diffusion approximations of empirical processes associated with some classes of discrete time Markov chains in population genetics. We refer to [16, 17, 20] for more details on the topic.

If $b = 0$, it has been proved in [1] that the closure of $(A, C^2(S^d))$ generates a bounded analytic semigroup, but to extend the result to the case of a non-vanishing drift, it is needed a careful estimate of the constants appearing in the study of the sectoriality of the one-dimensional operator

$$Au(x) = \gamma(x)x(1-x)u''(x) + b(x)u'(x), \qquad x \in [0,1], \tag{0.2}$$

where γ is a continuous strictly positive function and b is a continuous function such that $b(0) \geq 0$ and $b(1) \leq 0$. As already pointed out by Feller in the fifties (see [18, 19], see also [26, 11, 10]), these degenerate operators generate positive and contractive semigroups in $C([0,1])$ if suitable boundary conditions are added.

The aim of this paper is to present some results about generation, sectoriality and gradient estimates for the resolvent of suitable realizations of the operators

$$A^{\gamma,b}u(x) = \gamma x u''(x) + bu'(x), \tag{0.3}$$

with constants $\gamma > 0$ and $b \geq 0$, in the space $C([0,\infty])$, because they model in 0 the behaviour of the operators (0.2) near the end points 0 and 1.

To this end we review some results established mainly in [7, 22, 13, 14], and we propose them in a unified way. We point out that the works [7, 13, 14] are addressed mainly to the study of the operators (0.2), (0.3) in Hölder continuous function spaces, while we will

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mainly focus on spaces of continuous functions. Moreover, several proofs are different from those of the cited papers and could be of independent interest.

Precisely, we start considering the explicit expression of the kernel $p^{\gamma,b}(x,y,t)$, given, e.g., in [13, 7], of the solution operator $P_t^{\gamma,b}$ for the equation $\partial_t - A^{\gamma,b}$. After proving some estimates for $p^{\gamma,b}(x,y,t)$, we show that $(P_t^{\gamma,b})_{t\geq 0}$ is a C_0-semigroup in $C([0,\infty])$ and that its infinitesimal generator is $A^{\gamma,b}$ endowed with the domain

$$\begin{aligned} D(A^{\gamma,0}) &= \{u\in C([0,\infty])\cap C^2(]0,\infty[) \mid \lim_{x\to 0^+} A^{\gamma,0}u(x)=0, \\ &\quad \lim_{x\to+\infty} A^{\gamma,0}u(x)=0\}, \qquad \text{if } b=0, \\ D(A^{\gamma,b}) &= \{u\in C^1([0,\infty[)\cap C^2(]0,\infty[)\cap C([0,\infty]) \mid \\ &\quad \lim_{x\to 0^+} xu''(x)=0,\ \lim_{x\to+\infty} A^{\gamma,b}u(x)=0\}, \qquad \text{if } b>0. \end{aligned}$$

Moreover, we prove that the space of C^2-functions on $[0,\infty[$ that are constant in a neighbourhood of ∞ is a core for $(P_t^{\gamma,b})_{t\geq 0}$.

At this point, the analiticity of $(P_t^{\gamma,b})_{t\geq 0}$ in $C([0,\infty])$ follows immediately from the results in [22, 9], but with a careful analysis we also prove that, for any $B>0$ and $\gamma_0>0$ fixed, there exists a constant $C=C(B,\gamma_0)>0$ such that, for every $b\in[0,B]$ and $\gamma\geq\gamma_0$,

$$||tA^{\gamma,b}P_t^{\gamma,b}|| \leq C(B,\gamma_0), \quad t\geq 0,$$

that is, the analiticity constant is uniform in bounded intervals $[0,B]$ and in half-lines $[\gamma_0,\infty[$ with $\gamma_0>0$.

We also get pointwise gradient estimates both for the semigroup and for the resolvent $R(\lambda,A^{\gamma,b})$ and, in the case $b>0$, we prove that $\partial_x R(\lambda,A^{\gamma,b})$ is a continuous operator from $C([0,\infty])$ into itself and give an estimate of the operator norm.

The results presented in this paper play an important role in [4] to show the analyticity in spaces of continuous functions of the semigroup generated by some degenerate diffusion operators defined on domains of $\mathbb{R}^d$ with corners like (0.1). For further results on regularity in weighted L^p spaces of the semigroup generated by some classes of operators of type (0.1) we refer to [2, 3] and the references therein.

Notation. We will denote by $C_{\mathbf{b}}([0,\infty[)$ the space of continuous bounded functions on $[0,\infty[$ and by $C([0,\infty])$ the Banach space of continuous functions on $[0,\infty[$ converging at infinity, endowed with the sup-norm $||\cdot||_\infty$. Analogously, for every $k\in\mathbb{N}$, $C^k([0,\infty])$ stands for the space of functions $u\in C([0,\infty])$ with derivatives up to order k that have finite limits at ∞. Finally $C^k_c([0,\infty[)$ denotes the subspace of $C^k([0,\infty[)$ of functions with compact support and $C_0([0,\infty[)$ denotes the space of continuous functions on $[0,\infty[$ vanishing at ∞.

1. Preliminary results

Lemma 1.1. *Let $A>0$. Then there exists $C=C(A)>0$ such that, for every $0<a\leq A$ and $s>0$,*

$$\sum_{m=0}^{\infty}\frac{s^m}{m!\Gamma(m+a)} \leq C\frac{e^{2\sqrt{s}}}{s^{\frac{a}{2}-\frac{1}{4}}}\left(1+\frac{e^{\frac{C}{2\sqrt{s}}}}{\sqrt{s}}\right). \tag{1.1}$$

Proof. Recall that the Bessel modified function of the first kind and parameter $\nu\in\mathbb{R}$ is defined by the formula

$$I_\nu(x)=\sum_{m=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{\nu+2m}}{m!\Gamma(m+\nu+1)}, \quad x>0, \tag{1.2}$$

and so, for $\nu = a-1$ we have

$$\sum_{m=0}^{\infty} \frac{s^m}{m!\Gamma(m+a)} = \frac{1}{s^{\frac{a-1}{2}}} I_{a-1}(2\sqrt{s}), \quad s>0.$$

Moreover, by [23, (7.16)], for every $x>0$, the following estimate holds

$$\begin{aligned} |I_\nu(x)| &\le \frac{1}{(2\pi x)^{\frac12}} \left[e^x \left(|4\nu^2-1| + \pi^{\frac12} e^{\frac{|\nu^2-\frac14|}{x}} \frac{|(4\nu^2-1)(4\nu^2-9)|}{x} \right) \right. \\ &+ \left. e^{-x}\left(|4\nu^2-1| + 2e^{\frac{|\nu^2-\frac14|}{x}} \frac{|4\nu^2-1|}{x} \right) \right] \le \frac{C}{x^{\frac12}} e^x \left(1 + e^{\frac{C}{x}} \frac{1}{x} \right), \end{aligned}$$

where $C = \max\{\sqrt{\pi}|4\nu^2-1||4\nu^2-9|, 2|4\nu^2-1|\}$. Then the assertion follows by applying the previous estimate with $\nu = a-1$, $C = \max_{[0,A]}\{\sqrt{\pi}|4a^2-8a+3||4a^2-8a-5|, 2|4a^2-8a+3|\}$ and $x = 2\sqrt{s}$. $\square$

Lemma 1.2. *For every $\delta>0$ and for every $s>0$*

$$\sum_{m=0}^{\infty} |m-s| \frac{s^m}{(m+1)!} \le \delta \frac{e^s-1}{s} + \frac{1}{\delta}\left(e^s - 2 - s + \frac{e^s-1}{s} \right).$$

In particular, we have

$$\sum_{m=0}^{\infty} |m-s| \frac{s^m}{(m+1)!} = O(\sqrt{s}) \quad \text{as } s \to 0^+ \tag{1.3}$$

$$e^{-s} \sum_{m=0}^{\infty} |m-s| \frac{s^m}{(m+1)!} = O\left(\frac{1}{\sqrt{s}}\right) \quad \text{as } s \to \infty. \tag{1.4}$$

Proof. Fix any $\delta>0$. Then, for every $s>0$, we have

$$\begin{aligned} &\sum_{m=0}^{\infty} |m-s| \frac{s^m}{(m+1)!} = \sum_{|m-s|\le\delta} |m-s| \frac{s^m}{(m+1)!} + \sum_{|m-s|>\delta} |m-s| \frac{s^m}{(m+1)!} \\ \le\ & \delta \sum_{m=0}^{\infty} \frac{s^m}{(m+1)!} + \frac{1}{\delta} \sum_{m=0}^{\infty} (m-s)^2 \frac{s^m}{(m+1)!} \\ =\ & \delta \frac{e^s-1}{s} + \frac{1}{\delta}\left(\sum_{m=1}^{\infty} \frac{m^2 s^m}{(m+1)!} + s \sum_{m=0}^{\infty} \frac{s^{m+1}}{(m+1)!} - 2s \sum_{m=0}^{\infty} \frac{m s^m}{(m+1)!} \right) \\ =\ & \delta \frac{e^s-1}{s} + \frac{1}{\delta}\left(\sum_{m=1}^{\infty} \frac{m(m+1)s^m}{(m+1)!} + s(e^s-1) - 2s \sum_{m=0}^{\infty} \frac{m s^m}{(m+1)!} - \sum_{m=1}^{\infty} \frac{m s^m}{(m+1)!} \right) \\ =\ & \delta \frac{e^s-1}{s} + \\ & + \frac{1}{\delta}\left[s \sum_{m=1}^{\infty} \frac{s^{m-1}}{(m-1)!} + s(e^s-1) - (2s+1)\left(\sum_{m=1}^{\infty} \frac{(m+1)s^m}{(m+1)!} - \sum_{m=1}^{\infty} \frac{s^m}{(m+1)!} \right) \right] \\ =\ & \delta \frac{e^s-1}{s} + \frac{1}{\delta}\left[se^s + s(e^s-1) - (2s+1)\left(\sum_{m=1}^{\infty} \frac{s^m}{m!} - \frac{1}{s} \sum_{m=1}^{\infty} \frac{s^{m+1}}{(m+1)!} \right) \right] \\ =\ & \delta \frac{e^s-1}{s} + \frac{1}{\delta}\left(e^s - 2 - s + \frac{e^s-1}{s} \right). \end{aligned}$$

Choosing $\delta = \sqrt{s}$, we get that

$$\sum_{m=0}^{\infty} |m-s| \frac{s^m}{(m+1)!} \leq \frac{e^s-1}{\sqrt{s}} + \frac{1}{\sqrt{s}}\left(e^s - 2 - s + \frac{e^s-1}{s}\right).$$

Since

$$\frac{1}{\sqrt{s}}\left(e^s - 2 - s + \frac{e^s-1}{s}\right) \sim \frac{1}{2}\sqrt{s} \quad \text{and} \quad \frac{e^s-1}{\sqrt{s}} \sim \sqrt{s} \text{ as } s \to 0,$$

we easily deduce the assertion for $s \to 0^+$, while the behaviour as $s \to \infty$ follows observing that

$$\frac{e^s-1}{\sqrt{s}} + \frac{1}{\sqrt{s}}\left(e^s - 2 - s + \frac{e^s-1}{s}\right) \sim 2\frac{e^s}{\sqrt{s}} \text{ as } s \to \infty. \qquad \square$$

Lemma 1.3. *For every $s > 0$*

$$\frac{1}{\Gamma(s)} \int_0^{\infty} e^{-z} z^{s-1} |z-s| dz \leq 2\sqrt{s}.$$

Proof. For every $\delta > 0$ and $s > 0$

$$\begin{aligned}
& \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-z} z^{s-1} \,|z-s|\, dz \\
\leq\ & \frac{1}{\Gamma(s)} \int_{|z-s|<\delta} e^{-z} z^{s-1} \,|z-s|\, dz + \frac{1}{\Gamma(s)} \int_{|z-s|\geq\delta} e^{-z} z^{s-1} \,|z-s|\, dz \\
\leq\ & \frac{1}{\Gamma(s)} \left(\delta\Gamma(s) + \frac{1}{\delta}\int_0^{\infty} e^{-z} z^{s-1} \,(z-s)^2\, dz\right) \\
=\ & \frac{1}{\Gamma(s)} \left[\delta\Gamma(s) + \frac{1}{\delta}\left(\Gamma(s+2) - 2s\Gamma(s+1) + s^2\Gamma(s)\right)\right] \\
=\ & \frac{1}{\Gamma(s)} \left(\delta\Gamma(s) + \frac{1}{\delta}\Gamma(s+1)\right) = \delta + \frac{s}{\delta}.
\end{aligned}$$

Then the thesis follows by choosing $\delta = \sqrt{s}$. $\square$

2. The semigroup $(P_t^{\gamma,b})_{t\geq 0}$

Fix any $\gamma > 0$ and $b \geq 0$. For every $x, y \geq 0$ and $t > 0$ set

$$p^{\gamma,b}(x,y,t) = (\gamma t)^{-\frac{b}{\gamma}} e^{-\frac{x+y}{\gamma t}} y^{\frac{b}{\gamma}-1} \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\frac{b}{\gamma})} \frac{x^m y^m}{\gamma^{2m} t^{2m}} \qquad \text{if } b > 0$$

$$p^{\gamma,0}(x,y,t) = e^{-\frac{x}{\gamma t}} \delta_0(y) + (\gamma t)^{-1} e^{-\frac{x+y}{\gamma t}} \sum_{m=0}^{\infty} \frac{1}{(m+1)!m!} \frac{x^{m+1} y^m}{\gamma^{2m+1} t^{2m+1}} \qquad \text{if } b = 0,$$

where δ_0 is a point mass at 0. Thanks to (1.2) the kernels can be written as

$$p^{\gamma,b}(x,y,t) = \frac{1}{\gamma t}\left(\frac{x}{y}\right)^{\frac{1-b/\gamma}{2}} e^{-\frac{x+y}{\gamma t}} I_{b/\gamma-1}\left(\frac{2\sqrt{xy}}{\gamma t}\right) \qquad \text{if } b > 0 \tag{2.1}$$

$$p^{\gamma,0}(x,y,t) = e^{-\frac{x}{\gamma t}} \delta_0(y) + \frac{1}{\gamma t}\left(\frac{x}{y}\right)^{\frac{1}{2}} e^{-\frac{x+y}{\gamma t}} I_1\left(\frac{2\sqrt{xy}}{\gamma t}\right) \qquad \text{if } b = 0. \tag{2.2}$$

Therefore, we easily obtain that

$$\int_0^{\infty} p^{\gamma,b}(x,y,t) dy = 1, \qquad x \geq 0,\ t > 0. \tag{2.3}$$

Moreover, in [13, Corollary 9] (see also [7] for a stochastic approach), it is shown that

$$p^{\gamma,b}(x,y,t+s)=\int_0^\infty p^{\gamma,b}(x,z,t)p^{\gamma,b}(z,y,s)dz \qquad s,\, t>0,\ x,\, y\geq 0.$$

The properties above ensure that, for each $t>0$, the operator $P_t^{\gamma,b}\colon C_{\mathbf{b}}([0,\infty[)\to C_{\mathbf{b}}([0,\infty[)$ defined by

$$P_t^{\gamma,b}f(x)=\int_0^\infty p^{\gamma,b}(x,y,t)f(y)dy, \qquad f\in C_{\mathbf{b}}([0,\infty[),\ x\geq 0, \tag{2.4}$$

is well-defined and continuous and that the family $(P_t^{\gamma,b})_{t\geq 0}$ is a contraction semigroup in $C_{\mathbf{b}}([0,\infty[)$ (here, $P_0^{\gamma,b}:=I$). We also observe that, for every $f\in C_{\mathbf{b}}([0,\infty[)$ and $x\geq 0$, we have

$$P_t^{\gamma,b}f(x)=e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!\Gamma(m+\frac{b}{\gamma})}\int_0^\infty e^{-z}z^{m+\frac{b}{\gamma}-1}f(\gamma z t)dz \qquad \text{if } b>0, \tag{2.5}$$

while

$$P_t^{\gamma,0}f(x)=e^{-\frac{x}{\gamma t}}f(0)+e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^{m+1}\frac{1}{m!(m+1)!}\int_0^\infty e^{-z}z^m f(\gamma z t)dz \text{ if } b=0. \tag{2.6}$$

Since $\int_0^\infty p^{\gamma,b}(x,y,t)y^k dy$ is convergent for every $k\in\mathbb{N}$, with an abuse of notation we can set $P_t^{\gamma,b}(f)=\int_0^\infty p^{\gamma,b}(x,y,t)f(y)dy$ for every continuous function f with polynomial growth at infinity.

In the following, for each $x,\, y>0$ we set $\tau_x(y):=(y-x)$.

Lemma 2.1. *Let $b>0$ and $\gamma>0$. Then, for every $x,t>0$ and $k\in\mathbb{N}$, the following properties hold:*

(1) $P_t^{\gamma,b}(\tau_x)(x)=bt$,
(2) $P_t^{\gamma,b}(\tau_x^{k+1})(x)=-xP_t^{\gamma,b}(\tau_x^k)(x)+btP_t^{\gamma,b+\gamma}(\tau_x^k)(x)+xP_t^{\gamma,b+2\gamma}(\tau_x^k)(x)$.

In particular, for every $x,t>0$, we have

$$P_t^{\gamma,b}(\tau_x^2)(x)=2\gamma t x+t^2b(b+\gamma), \tag{2.7}$$
$$P_t^{\gamma,b}(|\tau_x|)(x)\leq\sqrt{2\gamma t x+t^2b(b+\gamma)}. \tag{2.8}$$

Proof. Let $b>0$. By (2.3) and (2.5) we obtain, for every $x,t>0$, that

$$\begin{aligned} P_t^{\gamma,b}(\tau_x)(x) &= e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!\Gamma(m+\frac{b}{\gamma})}\gamma t\int_0^\infty e^{-z}z^{m+\frac{b}{\gamma}}dz-x \\ &= e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{m+b/\gamma}{m!}\gamma t-x \\ &= x+bt-x=bt; \end{aligned}$$

hence, (1) is satisfied.

Fixed any $k \in \mathbb{N}$, we have, for every $x, t > 0$, that

$$\begin{aligned}
& P_t^{\gamma,b}(\tau_x^{k+1})(x) + xP_t^{\gamma,b}(\tau_x^k)(x) = \int_0^\infty p^{\gamma,b}(x,y,t)y(y-x)^k dy \\
= \;& \gamma t e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+\frac{b}{\gamma})} \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}}(\gamma t z - x)^k dz \\
= \;& \gamma t e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{m+\frac{b}{\gamma}}{m!\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}}(\gamma t z - x)^k dz \\
= \;& \gamma t e^{-\frac{x}{\gamma t}} \sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{(m-1)!\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}}(\gamma t z - x)^k dz \\
& + \gamma t \frac{b}{\gamma} e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}}(\gamma t z - x)^k dz \\
= \;& xP_t^{\gamma,b+2\gamma}(\tau_x^k)(x) + btP_t^{\gamma,b+\gamma}(\tau_x^k)(x)
\end{aligned}$$

and so (2) is also satisfied. As a consequence we deduce, for every $x, t > 0$, that

$$\begin{aligned}
P_t^{\gamma,b}(\tau_x^2)(x) &= -xP_t^{\gamma,b}(\tau_x)(x) + xP_t^{\gamma,b+2\gamma,}(\tau_x)(x) + btP_t^{\gamma,b+\gamma}(\tau_x)(x) \\
&= -xbt + x(b+2\gamma)t + bt(b+\gamma)t = 2\gamma t x + t^2 b(b+\gamma).
\end{aligned}$$

Finally, by applying Hölder inequality together with (2.3) and (2.7), we get, for every $x > 0$, that

$$\int_0^\infty p^{\gamma,b}(x,y,t)|x-y|dy \le \left(\int_0^\infty p^{\gamma,b}(x,y,t)|x-y|^2 dy\right)^{\frac{1}{2}} = \sqrt{2\gamma t x + t^2 b(b+\gamma)}. \qquad \square$$

Lemma 2.2. *Let $\gamma > 0$. Then, for every $x, t > 0$ the following properties hold:*

(1) $P_t^{\gamma,0}(\tau_x)(x) = 0$,
(2) $P_t^{\gamma,0}(\tau_x^2)(x) = 2\gamma t x$.

In particular, for every $x, t > 0$, we have

$$P_t^{\gamma,0}(|\tau_x|)(x) \le \sqrt{2\gamma t x}. \tag{2.9}$$

Proof. By (2.6) we obtain, for every $x, t > 0$, that

$$\begin{aligned}
P_t^{\gamma,0}(\tau_x)(x) &= -xe^{-\frac{x}{\gamma t}} + e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^{m+1} \frac{1}{m!(m+1)!} \int_0^\infty e^{-z} z^m (\gamma t z - x) dz \\
&= -xe^{-\frac{x}{\gamma t}} + \gamma t e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^{m+1} \frac{1}{m!} \\
&\quad -xe^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^{m+1} \frac{1}{(m+1)!} \\
&= -xe^{-\frac{x}{\gamma t}} + x - xe^{-\frac{x}{\gamma t}}(e^{\frac{x}{\gamma t}} - 1) \\
&= -xe^{-\frac{x}{\gamma t}} + x - x + xe^{-\frac{x}{\gamma t}} = 0;
\end{aligned}$$

hence, (1) is satisfied. Also, for every $x, t > 0$, we have

$$\begin{aligned}
P_t^{\gamma,0}(\tau_x^2)(x) &= x^2 e^{-\frac{x}{\gamma t}} + e^{-\frac{x}{\gamma t}} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m+1} \frac{1}{m!(m+1)!} \int_0^{\infty} e^{-z} z^m (\gamma t z - x)^2 dz \\
&= x^2 e^{-\frac{x}{\gamma t}} + \gamma^2 t^2 e^{-\frac{x}{\gamma t}} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m+1} \frac{(m+2)!}{m!(m+1)!} \\
&\quad -2\gamma t x e^{-\frac{x}{\gamma t}} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m+1} \frac{(m+1)!}{m!(m+1)!} + x^2 e^{-\frac{x}{\gamma t}} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m+1} \frac{m!}{m!(m+1)!} \\
&= x^2 e^{-\frac{x}{\gamma t}} + \gamma^2 t^2 e^{-\frac{x}{\gamma t}} \frac{x^2}{\gamma^2 t^2} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m} \frac{1}{m!} + 2\gamma^2 t^2 e^{-\frac{x}{\gamma t}} \frac{x}{\gamma t} \sum_{m=0}^{\infty} \left(\frac{x}{\gamma t}\right)^{m} \frac{1}{m!} \\
&\quad -2\gamma t x e^{-\frac{x}{\gamma t}} \frac{x}{\gamma t} e^{\frac{x}{\gamma t}} + x^2 e^{-\frac{x}{\gamma t}} (e^{\frac{x}{\gamma t}} - 1) \\
&= x^2 e^{-\frac{x}{\gamma t}} + x^2 + 2\gamma t x - 2x^2 + x^2 - x^2 e^{-\frac{x}{\gamma t}} = 2\gamma t x
\end{aligned}$$

and so (2) is satisfied. Finally, by applying Hölder inequality together with (2.3) and the property (2) above one easily shows (2.9). □

Lemma 2.3. *Let $b \geq 0$ and $\gamma > 0$. Then the following properties hold.*

(1) *If $f \in C_{\mathbf{b}}([0,\infty[)$, then*

$$\lim_{t \to 0^+} P_t^{\gamma,b} f = f$$

uniformly on compact subsets of $[0,\infty[$.

(2) *If $b > 0$, then there exists a constant $C = C(b) > 0$ such that, for every $f \in C_c(\mathbb{R}^+)$ with supp$(f) \subseteq [0,M]$ and $x, t > 0$, we have*

$$|P_t^{\gamma,b} f(x)| \leq C ||f||_\infty e^{-\frac{x - 2\sqrt{xM}}{\gamma t}} \left(\frac{M}{x}\right)^{\frac{b}{2\gamma} + \frac{1}{4}} \sqrt{\frac{\gamma t}{M}} \left(1 + e^{\frac{C\gamma t}{2\sqrt{xM}}} \frac{\gamma t}{\sqrt{xM}}\right). \tag{2.10}$$

If $b = 0$, then there exists a constant $C > 0$ such that, for every $f \in C_c(\mathbb{R}^+)$ with supp$(f) \subseteq [0,M]$ and $x, t > 0$, we have

$$|\P_t^{\gamma,0} f(x)| \leq C ||f||_\infty e^{-\frac{x - 2\sqrt{xM}}{\gamma t}} \left(\frac{M}{x}\right)^{\frac{1}{4}} \sqrt{\frac{\gamma t}{M}} \left(1 + e^{\frac{C\gamma t}{2\sqrt{xM}}} \frac{\gamma t}{\sqrt{xM}}\right). \tag{2.11}$$

Therefore, for every $t > 0$, $\lim_{x \to \infty} P_t^{\gamma,b} f(x) = 0$ and $\lim_{t \to 0^+} P_t^{\gamma,b} f(x) = f(x)$ uniformly on $[0,+\infty[$.

Proof. (1) Let $f \in C_{\mathbf{b}}([0,\infty[)$ and let $M > 0$. We prove that $\lim_{t\to 0} P_t^{\gamma,b} f(x) = f(x)$ uniformly in $[0,M]$. Indeed, let $\varepsilon > 0$ and let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0,M]$ satisfy $|x - y| < \delta$. Then, by (2.8) and (2.9), we obtain, for every $x \in [0,M]$

and $t>0$, that

$$\begin{aligned}
|P_t^{\gamma,b}f(x)-f(x)| &= \left|\int_0^\infty p^{\gamma,b}(x,y,t)(f(y)-f(x))dy\right| \\
&\le \varepsilon\int_{|x-y|<\delta} p^{\gamma,b}(x,y,t)dy + 2||f||_\infty \int_{|x-y|\ge\delta} p^{\gamma,b}(x,y,t)dy \\
&\le \varepsilon + \frac{2||f||_\infty}{\delta}\int_0^\infty p^{\gamma,b}(x,y,t)|x-y|dy \\
&\le \varepsilon + \frac{2||f||_\infty}{\delta}\sqrt{2\gamma t x + t^2 b\,(b+\gamma)} \\
&\le \varepsilon + \frac{2||f||_\infty}{\delta}\sqrt{2\gamma t M + t^2 b\,(b+\gamma)}.
\end{aligned}$$

We now get immediately the assertion.

(2) Let $f\in C_c([0,\infty[)$ with $\mathrm{supp}(f)\subseteq[0,M]$. If $b>0$, then, by (2.5) and Lemma 1.1, we obtain, for every $x,t>0$, that

$$\begin{aligned}
|P_t^{\gamma,b}f(x)| &\le ||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+b/\gamma)}\int_0^{\frac{M}{\gamma t}} e^{-z}z^{m+b/\gamma-1}dz \\
&\le ||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+b/\gamma)}\frac{1}{m+b/\gamma}\left(\frac{M}{\gamma t}\right)^{m+b/\gamma} \\
&= ||f||_\infty e^{-\frac{x}{\gamma t}}\left(\frac{M}{\gamma t}\right)^{b/\gamma}\sum_{m=0}^\infty\left(\frac{xM}{\gamma^2t^2}\right)^m \frac{1}{m!\Gamma(m+b/\gamma+1)} \\
&\le C||f||_\infty e^{-\frac{x}{\gamma t}}\left(\frac{M}{\gamma t}\right)^{b/\gamma} e^{\frac{2\sqrt{xM}}{\gamma t}}\frac{1}{\left(\frac{xM}{\gamma^2t^2}\right)^{\frac{b}{2\gamma}+\frac14}}\left(1+e^{\frac{C\gamma t}{2\sqrt{xM}}}\frac{\gamma t}{\sqrt{xM}}\right) \\
&= C||f||_\infty e^{-\frac{x-2\sqrt{xM}}{\gamma t}}\sqrt{\frac{\gamma t}{M}}\left(\frac{M}{x}\right)^{\frac{b}{2\gamma}+\frac14}\left(1+e^{\frac{C\gamma t}{2\sqrt{xM}}}\frac{\gamma t}{\sqrt{xM}}\right).
\end{aligned}$$

If $b=0$, then, by (2.6) and Lemma 1.1, we obtain, for every $x,t>0$, that

$$\begin{aligned}
|P_t^{\gamma,0}f(x)| &\le ||f||_\infty e^{-\frac{x}{\gamma t}} + ||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^{m+1}\frac{1}{m!(m+1)!}\int_0^{\frac{M}{\gamma t}} e^{-z}z^m dz \\
&\le ||f||_\infty e^{-\frac{x}{\gamma t}}\left(1+\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^{m+1}\frac{1}{(m+1)!(m+1)!}\left(\frac{M}{\gamma t}\right)^{m+1}\right) \\
&= ||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{xM}{\gamma^2t^2}\right)^m\frac{1}{m\Gamma(m+1)} \\
&\le C||f||_\infty e^{-\frac{x}{\gamma t}} e^{\frac{2\sqrt{xM}}{\gamma t}}\frac{1}{\left(\frac{xM}{\gamma^2t^2}\right)^{\frac14}}\left(1+e^{\frac{C\gamma t}{2\sqrt{xM}}}\frac{\gamma t}{\sqrt{xM}}\right) \\
&= C||f||_\infty e^{-\frac{x-2\sqrt{xM}}{\gamma t}}\sqrt[4]{\frac{M}{x}}\sqrt{\frac{\gamma t}{M}}\left(1+e^{\frac{C\gamma t}{2\sqrt{xM}}}\frac{\gamma t}{\sqrt{xM}}\right).
\end{aligned}$$

Now, if $f\in C_c([0,\infty[)$ with $\mathrm{supp}(f)\subseteq[0,M]$, then (2.10) ((2.11) for $b=0$) clearly implies that $\lim_{x\to\infty}P_t^{\gamma,b}f(x)=0$ for every $t>0$. On the other hand, fixed any $\varepsilon>0$, by (2.10) ((2.11) for $b=0$) there exists $N>M$ so that $|P_t^{\gamma,b}f(x)|<\varepsilon/2$ for every $x>N$ and

$0 < t \le 1$. By (1) there is also $\overline{t} \in]0,1]$ for which $\max_{x\in[0,N]} |P_t^{\gamma,b} f(x) - f(x)| < \varepsilon/2$ for every $0 < t < \overline{t}$. So, it follows, for every $0 < t < \overline{t}$, that

$$\sup_{x\in[0,\infty[} |P_t^{\gamma,b} f(x) - f(x)| \le \max_{x\in[0,N]} |P_t^{\gamma,b} f(x) - f(x)| + \sup_{x>N} |P_t^{\gamma,b} f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes the proof. □

Proposition 2.4. *Let $b \ge 0$ and $\gamma > 0$. Then $(P_t^{\gamma,b})_{t\ge0}$ is a C_0-semigroup in $C([0,\infty])$.*

Proof. Since each operator $P_t^{\gamma,b}$ preserves constant functions and is contractive, it suffices to prove that $(P_t^{\gamma,b})_{t\ge0}$ is a C_0-semigroup in $C_0([0,\infty[)$. So, we first observe that Lemma 2.3(2) and the fact that $P_t^{\gamma,b}$ is a continuous linear operator from $C_{\mathbf{b}}([0,\infty[)$ into itself ensure that

$$P_t^{\gamma,b}(C_0([0,\infty[)) = P_t^{\gamma,b}(\overline{C_c([0,\infty[)}) \subseteq \overline{P_t^{\gamma,b}(C_c([0,\infty[))} \subseteq C_0([0,\infty[).$$

Hence, $P_t^{\gamma,b}$ is a well-defined bounded linear operator from $C_0([0,\infty[)$ into itself. On the other hand, if $f \in C_c([0,\infty[)$, then by Lemma 2.3(2) we have $\lim_{t\to0^+} P_t^{\gamma,b} f = f$ uniformly on $[0,+\infty[$. The density of $C_c([0,\infty[)$ in $C_0([0,\infty[$ and the contractivity of $(P_t^{\gamma,b})_{t\ge0}$ imply that $\lim_{t\to0^+} P_t^{\gamma,b} f = f$ uniformly on $[0,+\infty[$ for every $f \in C_0([0,\infty[)$. This completes the proof. □

In the sequel, we denote by $(A, D(A))$ the generator of $(P_t^{\gamma,b})_{t\ge0}$ in $C([0,\infty])$ and set

$$D := \{C^2([0,\infty]) \mid f \text{ is constant in a neighbourhood of } +\infty\}. \tag{2.12}$$

Proposition 2.5. *Let $b \ge 0$, $\gamma > 0$ and let D be defined according to* (2.12). *Then $D \subseteq D(A)$ and*

$$Af(x) = \gamma x f''(x) + bf'(x)$$

for every $f \in D$ and $x \ge 0$.

Proof. Fix any $f \in D$. Since each $P_t^{\gamma,b}$ preserves constant functions, we can assume w.l.o.g. that $f \in C^2([0,\infty[)$ with $\mathrm{supp}(f) \subseteq [0,M]$.

Now, let $b > 0$. Then by (2.10) we have, for every $x > 9M$ and $t > 0$, that

$$\begin{aligned} &\left| \frac{P_t^{\gamma,b} f(x) - f(x)}{t} \right| \le \\ \le\ & \frac{C||f||_\infty}{t} e^{-\frac{x-2\sqrt{xM}}{\gamma t}} \left(\frac{M}{x}\right)^{\frac{b}{2\gamma}+\frac14} \sqrt{\frac{\gamma t}{M}} \left(1 + e^{\frac{C\gamma t}{2\sqrt{xM}}} \frac{\gamma t}{\sqrt{xM}}\right) \\ \le\ & \frac{C||f||_\infty}{t} e^{-\frac{3M}{\gamma t}} \sqrt{\frac{\gamma t}{M}} \left(1 + e^{\frac{C\gamma t}{M}} \frac{\gamma t}{M}\right). \end{aligned}$$

It follows that

$$\lim_{t\to0^+} \frac{P_t^{\gamma,b} f(x) - f(x)}{t} = 0 = \gamma x f''(x) + bf'(x) \text{ uniformly on } [9M,\infty[. \tag{2.13}$$

On the other hand, for every $x, y \ge 0$ we can write $f(y) = f(x) + f'(x)\tau_x(y) + \frac12 f''(x)\tau_x^2(y) + \omega(x,y)$, where $\omega(x,y) = \frac{f''(\xi)-f''(x)}{2}(x-y)^2$ with ξ belonging to the interval having x and

y as endpoints. Since $|\omega(x,y)| \le ||f''||_\infty (x-y)^2$ for every $x, y \ge 0$, by (2.7) we obtain, for every $x, t > 0$, that

$$\begin{aligned}
&\left| \frac{P_t^{\gamma,b} f(x) - f(x)}{t} - \gamma x f''(x) - b f'(x) \right| \\
= \;& |\frac{1}{2} f''(x) t b(b+\gamma) + \int_0^\infty p^{\gamma,b}(x,y,t)\omega(x,y)dy| \\
\le \;& \frac{1}{2} t ||f''||_\infty b(b+\gamma) + ||f''||_\infty \int_0^\infty p^{\gamma,b}(x,y,t)(y-x)^2 dy \\
= \;& \frac{1}{2} t ||f''||_\infty b(b+\gamma) + ||f''||_\infty P_t^{\gamma,b}(\tau_x^2)(x) \\
= \;& \frac{1}{2} t ||f''||_\infty b(b+\gamma) + ||f''||_\infty [2\gamma t x + t^2 b(b+\gamma)].
\end{aligned}$$

It follows that

$$\lim_{t\to 0^+} \frac{P_t^{\gamma,b} f(x) - f(x)}{t} = \gamma x f''(x) + b f'(x) \text{ uniformly on compact subsets of } [0,\infty[. \quad (2.14)$$

So, by (2.13) and (2.14) we get that $\lim_{t\to 0^+} \frac{P_t^{\gamma,b} f(x) - f(x)}{t} = \gamma x f''(x) + b f'(x)$ in $C([0,\infty])$.

In case $b = 0$ the proof is analogous via (2.11) and Lemma 2.2(2). □

3. The infinitesimal generator $A^{\gamma,b}$

We now consider the operator $A^{\gamma,b} u = \gamma x u'' + b u'$, with the domain $D(A^{\gamma,b})$ defined in the introduction, i.e.,

$$\begin{aligned}
D(A^{\gamma,0}) \;&=\; \{u \in C([0,\infty]) \cap C^2(]0,\infty[) \mid \lim_{x\to 0^+} A^{\gamma,0}u(x) = 0, \\
&\quad \lim_{x\to+\infty} A^{\gamma,0}u(x) = 0\}, \qquad \text{if } b = 0, \\
D(A^{\gamma,b}) \;&=\; \{u \in C^1([0,\infty[) \cap C^2(]0,\infty[) \cap C([0,\infty]) \mid \\
&\quad \lim_{x\to 0^+} x u''(x) = 0, \ \lim_{x\to+\infty} A^{\gamma,b}u(x) = 0\}, \qquad \text{if } b > 0.
\end{aligned}$$

It is known that $(A^{\gamma,b}, D(A^{\gamma,b}))$ generates a C_0-contractive semigroup in $C([0,\infty])$ (see, e.g., [11, 26]).

Proposition 3.1. *Let $b \ge 0$, $\gamma > 0$ and let D be defined according to* (2.12)*. Then the following properties hold.*

(1) $D(A^{\gamma,b}) \subseteq \{u \in C^2(]0,\infty[) \mid \lim_{x\to 0} x u'(x) = 0, \ \lim_{x\to\infty} u'(x) = 0\}$.
(2) *D is a core for $(A^{\gamma,b}, D(A^{\gamma,b}))$.*

Proof. (1) Let $u \in D(A^{\gamma,b})$. Then, for every $\varepsilon > 0$ there exists $M > 0$ such that $|A^{\gamma,b}u(x)| < \gamma\varepsilon$ for every $x > M$. On the other hand, we have, for every $x > M$, that

$$\begin{aligned}
\int_M^x A^{\gamma,b} u(s) ds \;&=\; \gamma x u'(x) - \gamma M u'(M) + \int_M^x (b-\gamma) u'(s) ds \\
&=\; \gamma x u'(x) - \gamma M u'(M) + (b-\gamma)(u(x) - u(M))
\end{aligned}$$

and hence,

$$u'(x) = \frac{1}{\gamma x} \int_M^x A^{\gamma,b} u(s) ds + \frac{M}{x} u'(M) + \frac{b-\gamma}{\gamma x}(u(M) - u(x)).$$

So, we deduce, for every $x > M$, that

$$\begin{aligned}|u'(x)| &\leq \frac{1}{\gamma x}\int_M^x |A^{\gamma,b}u(s)|ds + \frac{1}{\gamma x}\left[\gamma M|u'(M)| + |(b-\gamma)(u(M)-u(x)|\right] \\ &\leq \varepsilon + \frac{1}{\gamma x}\left[\gamma M|u'(M)| + |(b-\gamma)(u(M)-u(x))|\right].\end{aligned}$$

This ensures that $\limsup_{x\to+\infty}|u'(x)| \leq \varepsilon$; it follows that $\lim_{x\to+\infty} u'(x) = 0$ as ε is arbitrary.

If $b > 0$, then clearly $\lim_{x\to 0^+} xu'(x) = 0$. If $b = 0$, we can apply the same argument as before integrating on a suitable small interval $[0,\delta]$.

(2) Let $u \in D(A^{\gamma,b})$ and let $\phi \in C_c([0,\infty[)$ satisfy $\phi(x) = 0$ if $x > 2$ and $\phi(x) = 1$ if $x < 1$. For each $n \in \mathbb{N}$ we set

$$u_n(x) := \begin{cases} u(\frac{1}{n}) + (x-\frac{1}{n})u'(\frac{1}{n}) + \frac{1}{2}(x-\frac{1}{n})^2u''(\frac{1}{n}) & 0 \leq x \leq \frac{1}{n} \\ u(x) & \frac{1}{n} \leq x \leq 1 \\ (u(x)-l)\phi\left(\frac{x}{n}\right) + l & x \geq 1, \end{cases}$$

where $l := \lim_{x\to\infty} u(x)$. Then $u_n \in C^2([0,\infty[) \cap C([0,\infty])$ and u_n is constant in a neighborhood of $+\infty$ as it is easy to verify. So, $(u_n)_n \subset D$. It is also straightforward to prove that $u_n \to u$ uniformly on $[1,+\infty[$. Moreover, for every $n \in \mathbb{N}$, we have

$$\sup_{x\in[0,1]}|u_n(x)-u(x)| \leq \sup_{0\leq x\leq\frac{1}{n}}\left|u(x) - u\left(\frac{1}{n}\right)\right| + \frac{1}{n}\left|u'\left(\frac{1}{n}\right)\right| + \frac{1}{n^2}\left|u''\left(\frac{1}{n}\right)\right|.$$

Since $\lim_{x\to 0^+} xu'(x) = 0$ by (1), it follows that $u_n \to u$ uniformly on $[0,1]$.

Therefore, $u_n \to u$ uniformly on $[0,\infty[$.

On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}&\sup_{x\in[0,1]}|A^{\gamma,b}u_n(x) - A^{\gamma,b}u(x)| = \\ &= \sup_{0\leq x\leq\frac{1}{n}}\left|\gamma x u''\left(\frac{1}{n}\right) + b\left[u'\left(\frac{1}{n}\right) + \left(x-\frac{1}{n}\right)u''\left(\frac{1}{n}\right)\right] - \gamma x u''(x) - bu'(x)\right| \\ &\leq \sup_{0\leq x\leq\frac{1}{n}}\left(|\gamma x u''(x)| + \frac{\gamma+b}{n}\left|u''\left(\frac{1}{n}\right)\right| + b\left|u'(x) - u'\left(\frac{1}{n}\right)\right|\right)\end{aligned}$$

(observe that the term $b\left|u'(x) - u'\left(\frac{1}{n}\right)\right|$ disappears in the above inequality for $b = 0$) and

$$\begin{aligned}&\sup_{x\geq 1}|A^{\gamma,b}u_n(x) - A^{\gamma,b}u(x)| = \\ &= \sup_{x\geq 1}\left|\left(\phi\left(\frac{x}{n}\right)-1\right)A^{\gamma,b}u(x) + 2\frac{\gamma x}{n}u'(x)\phi'\left(\frac{x}{n}\right) + \left[\frac{\gamma x}{n^2}\phi''\left(\frac{x}{n}\right) + \frac{b}{n}\phi'\left(\frac{x}{n}\right)\right](u(x)-l)\right| \\ &\leq \sup_{x\geq n}\left|\left(\phi\left(\frac{x}{n}\right)-1\right)A_{\gamma,b}u(x)\right| \\ &\quad + \sup_{n\leq x\leq 2n}\left|\frac{2\gamma x}{n}u'(x)\phi'\left(\frac{x}{n}\right) + \left[\frac{\gamma x}{n^2}\phi''\left(\frac{x}{n}\right) + \frac{b}{n}\phi'\left(\frac{x}{n}\right)\right](u(x)-l)\right| \\ &\leq \sup_{x\geq n}\left|A^{\gamma,b}u(x)\right| + 4||\phi'||_\infty\gamma\sup_{n\leq x\leq 2n}|u'(x)| + \left(\frac{2\gamma}{n}||\phi''||_\infty + \frac{b}{n}||\phi'||_\infty\right)\sup_{n\leq x\leq 2n}|u(x)-l|.\end{aligned}$$

Taking (1) into account, it follows that $A^{\gamma,b}u_n \to A^{\gamma,b}u$ uniformly on $[0,+\infty[$.

Therefore, D is a core for $(A^{\gamma,b}, D(A^{\gamma,b}))$. □

Remark 3.2. For similar results in a more general setting we refer to [5].

Proposition 3.3. *Let $b \geq 0$ and $\gamma > 0$. Then $(A^{\gamma,b}, D(A^{\gamma,b}))$ is the infinitesimal generator of $(P_t^{\gamma,b})_{t\geq 0}$.*

Proof. By Propositions 2.4 and 3.1(2) we have that $D \subseteq D(A)$, A and $A^{\gamma,b}$ coincide on D and that $(\lambda I - A)(D) = (\lambda I - A^{\gamma,b})(D)$ is dense in $C([0,\infty])$ for some $\lambda > 0$. It follows that D is also a core for $(A, D(A))$. Since $(A, D(A))$ and $(A^{\gamma,b}, D(A^{\gamma,b}))$ are closed operators in $C([0,\infty])$, the thesis follows.

A straightforward application of the First Trotter-Kato Approximation theorem (see, e.g., [12, Chap. III, Theorem 4.8]) and of the results above gives the following result.

Corollary 3.4. *Let $b \geq 0$ and $\gamma > 0$. Then, for every $f \in C([0,\infty])$,*

$$\lim_{b\to 0^+} P_t^{\gamma,b} f = P_t^{\gamma,0} f$$

in $C([0,\infty])$ uniformly for t in compact intervals.

4. Analyticity constants for $(P_t^{\gamma,b})_{t\geq 0}$ in $C([0,\infty])$

It is known that, for every $b \geq 0$ and $\gamma > 0$, the operator $(A^{\gamma,b}, D(A^{\gamma,b}))$ generates a bounded analytic C_0–semigroup of angle $\pi/2$ in $C([0,\infty])$. Indeed, $A^{\gamma,b}u$ has the same behaviour in 0 of the operator $B^{\gamma,b}$, defined by $B^{\gamma,b}u = xu''(x) + bu'(x)$, with domain

$$D(B^{\gamma,0}) = \{u \in C([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0} Bu = 0,\ u'(1) = 0\} \qquad if\ b = 0,$$
$$D(B^{\gamma,b}) = \{u \in C^1([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0} xu''(x) = 0,\ u'(1) = 0\} \qquad if\ b > 0,$$

which generates a bounded analytic C_0–semigroup of angle $\pi/2$ in $C([0,1])$, see [22, 9]. On the other hand, by performing the change of variable $x = \frac{1}{y}$, it can be seen that $A^{\gamma,b}$ behaves at ∞ as the operator $C^{\gamma,b}$, defined by $C^{\gamma,b}v = \gamma y^3 v'' + (2-b)y^2 v'$, with domain

$$D(C^{\gamma,b}) = \{u \in C([0,1]) \cap C^2(]0,1[) \mid \lim_{x\to 0^+} C^{\gamma,b}u = 0,\ u'(1) = 0\}$$

behaves near 0. By [10, Theorem 4.20] and the comments below, $(C^{\gamma,b}, D(C^{\gamma,b}))$ generates a bounded analytic C_0–semigroup of angle $\pi/2$ in $C([0,1])$.

Now, by suitable cut and paste tecniques (see, e.g., [9, Proposition 2.4]), it easily follows that $A^{\gamma,b}$ generates a bounded analytic C_0–semigroup of angle $\pi/2$ in $C([0,\infty])$. So, there exists $M = M(\gamma,b) > 0$ such that $||tA^{\gamma,b}P_t^{\gamma,b}|| \leq M_{\gamma,b}$ for every $t \geq 0$. Nevertheless, we now show that $M(\gamma,b)$ is uniformly bounded in bounded intervals $[0,B]$ and in half–lines $[\gamma_0,\infty[$ with $B, \gamma_0 > 0$.

Proposition 4.1. *Let $B, \gamma_0 > 0$. Then, for every $b \in [0,B]$, $\gamma \geq \gamma_0$ and $f \in C([0,\infty])$,*

$$||tA^{\gamma,b}P_t^{\gamma,b}f||_\infty \leq \frac{2(1+\sqrt{2}+b)}{\gamma}||f||_\infty \leq \frac{2(1+\sqrt{2}+B)}{\gamma_0}||f||_\infty, \qquad t \geq 0. \tag{4.1}$$

Proof. Fix $b > 0$ and $\gamma > 0$. If $f \in C([0,\infty])$, then straightforward calculations (see, e.g., [7, Lemma 4.5]) show, for every $x \geq 0$ and $t > 0$, that

$$(P_t^{\gamma,b}f)'(x) = \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\sum_{m=0}^{\infty}\left(\frac{x}{\gamma t}\right)^m \frac{1}{m!}\int_0^\infty e^{-z}\left(\frac{z^{m+\frac{b}{\gamma}}}{\Gamma(m+\frac{b}{\gamma}+1)} - \frac{z^{m+\frac{b}{\gamma}-1}}{\Gamma(m+\frac{b}{\gamma})}\right) \times f(\gamma z t)dz, \tag{4.2}$$

$$(P_t^{\gamma,b}f)''(x) = \frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\sum_{m=1}^{\infty}\left(\frac{x}{\gamma t}\right)^{m-1}\frac{1}{(m-1)!} \tag{4.3}$$

$$\times\int_0^{\infty}e^{-z}\left(\frac{z^{m+\frac{b}{\gamma}}}{\Gamma(m+\frac{b}{\gamma}+1)}-2\frac{z^{m+\frac{b}{\gamma}-1}}{\Gamma(m+\frac{b}{\gamma})}+\frac{z^{m+\frac{b}{\gamma}-2}}{\Gamma(m+\frac{b}{\gamma}-1)}\right)f(\gamma zt)\frac{dz}{\gamma t}.$$

So, by Lemma 1.3 we obtain, for every $x \geq 0$ and $t > 0$, that

$$\begin{aligned}|(P_t^{\gamma,b}f)'(x)| &\leq ||f||_\infty\frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\sum_{m=0}^{\infty}\left(\frac{x}{\gamma t}\right)^{m}\frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)}\times\\ &\times\int_0^{\infty}e^{-z}z^{m+\frac{b}{\gamma}-1}\left|z-m-\frac{b}{\gamma}\right|dz\\ \leq\ & ||f||_\infty\frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\sum_{m=0}^{\infty}\left(\frac{x}{\gamma t}\right)^{m}\frac{2}{m!\sqrt{m+\frac{b}{\gamma}}}\leq ||f||_\infty\frac{2}{\gamma t}.\end{aligned} \tag{4.4}$$

On the other hand, summing by parts in (4.3) we have, for every $x \geq 0$ and $t > 0$, that

$$\begin{aligned}x(P_t^{\gamma,b}f)''(x) &= -\frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\frac{x}{\gamma t}\int_0^{\infty}e^{-z}f(\gamma zt)\left[\frac{z^{b/\gamma}}{\Gamma(\frac{b}{\gamma}+1)}-\frac{z^{b/\gamma-1}}{\Gamma(\frac{b}{\gamma})}\right]dz+\\ &+\frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\sum_{m=1}^{\infty}\left(\frac{x}{\gamma t}\right)^{m}\frac{1}{m!}\left(m-\frac{x}{\gamma t}\right)\int_0^{\infty}e^{-z}f(\gamma zt)\left[\frac{z^{m+b/\gamma}}{\Gamma(m+\frac{b}{\gamma}+1)}-\frac{z^{m+b/\gamma-1}}{\Gamma(m+\frac{b}{\gamma})}\right]dz.\end{aligned}$$

Therefore, again by Lemma 1.3 we deduce, for every $x \geq 0$ and $t > 0$, that

$$\begin{aligned}|x(P_t^{\gamma,b}f)''(x)| &\leq \frac{2}{\gamma t}||f||_\infty+\\ &+\frac{1}{\gamma t}||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=1}^{\infty}\left(\frac{x}{\gamma t}\right)^{m}\frac{1}{m!}\left|m-\frac{x}{\gamma t}\right|\int_0^{\infty}\frac{z^{m+b/\gamma-1}}{\Gamma(m+\frac{b}{\gamma}+1)}\left|z-m-\frac{b}{\gamma}\right|dz\\ &\leq \frac{2}{\gamma t}||f||_\infty+\frac{1}{\gamma t}||f||_\infty e^{-\frac{x}{\gamma t}}\sum_{m=1}^{\infty}\left(\frac{x}{\gamma t}\right)^{m}\frac{1}{m!}\left|m-\frac{x}{\gamma t}\right|\frac{2}{\sqrt{m}}\\ &\leq \frac{2(1+\sqrt{2})}{\gamma t}||f||_\infty,\end{aligned} \tag{4.5}$$

as by Hölder inequality we have

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{s^m}{m!}|s-m|m^{-\frac{1}{2}} &\leq \left(\sum_{m=1}^{\infty}\frac{s^m}{m!}[s-m]^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\frac{s^m}{m!m}\right)^{\frac{1}{2}}\\ &\leq \sqrt{e^s s}\sqrt{\frac{2e^s}{s}}=\sqrt{2}e^s, \quad s>0.\end{aligned} \tag{4.6}$$

Now, let $b=0$. If $f\in C([0,\infty])$, then straightforward calculations (see, e.g., [7, Lemma 4.1]) show, for every $x\geq 0$ and $t>0$, that

$$\begin{aligned}(P_t^{\gamma,0}f)'(x) &= e^{-\frac{x}{\gamma t}}\int_0^\infty [f(\gamma zt)-f(0)]e^{-z}\frac{dz}{\gamma t} \\ &\quad +e^{-\frac{x}{\gamma t}}\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!}\int_0^\infty f(\gamma zt)e^{-z}\left[\frac{z^m}{m!}-\frac{z^{m-1}}{(m-1)!}\right]\frac{dz}{\gamma t}, \end{aligned} \tag{4.7}$$

$$\begin{aligned}(P_t^{\gamma,0}f)''(x) &= \frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\int_0^\infty [f(\gamma zt)-f(0)]e^{-z}(z-2)\frac{dz}{\gamma t} \\ &\quad +\frac{1}{\gamma t}e^{-\frac{x}{\gamma t}}\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!}\int_0^\infty f(\gamma zt)e^{-z}\left[\frac{z^{m+1}}{(m+1)!}-2\frac{z^m}{m!}+\frac{z^{m-1}}{(m-1)!}\right]\frac{dz}{\gamma t}. \end{aligned} \tag{4.8}$$

Since $P_t^{\gamma,0}g = P_t^{\gamma,0}f - f(0)$ if $g=f-f(0)$ and so $(P_t^{\gamma,0}g)'=(P_t^{\gamma,0}f)'$ and $(P_t^{\gamma,0}g)''=(P_t^{\gamma,0}f)''$, w.l.o.g. we may suppose $f(0)=0$. Therefore, summing by parts in (4.8) we have, for every $x\geq 0$ and $t>0$, that

$$\begin{aligned}x(P_t^{\gamma,0}f)''(x) &= -e^{-\frac{x}{\gamma t}}\frac{x}{\gamma t}\frac{1}{\gamma t}\int_0^\infty f(\gamma zt)e^{-z}dz+ \\ &+e^{-\frac{x}{\gamma t}}\frac{1}{\gamma t}\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m\frac{1}{m!}\left(\frac{x}{\gamma t}-m\right)\int_0^\infty f(\gamma zt)e^{-z}\frac{z^{m-1}}{m!}(m-z)dz.\end{aligned}$$

By Lemma 1.3 and (4.6) we obtain, for every $x\geq 0$ and $t>0$, that

$$\begin{aligned}|x(P_t^{\gamma,0}f)''(x)| &\leq \frac{||f||_\infty}{\gamma t}e^{-\frac{x}{\gamma t}}\frac{x}{\gamma t} \\ &+e^{-\frac{x}{\gamma t}}\frac{1}{\gamma t}\sum_{m=1}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!}\left|\frac{x}{\gamma t}-m\right|\int_0^\infty f(\gamma zt)e^{-z}\frac{z^{m-1}}{m!}|m-z|dz \\ &\leq \frac{||f||_\infty}{\gamma t}e^{-\frac{x}{\gamma t}}\left[\frac{x}{\gamma t}+2\sum_{m=1}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!\sqrt{m}}\left|\frac{x}{\gamma t}-m\right|\right] \\ &\leq \frac{||f||_\infty}{\gamma t}e^{-\frac{x}{\gamma t}}\left[\frac{x}{\gamma t}+2\sqrt{2}e^{-\frac{x}{\gamma t}}\right]\leq \frac{1+2\sqrt{2}}{\gamma t}||f||_\infty. \end{aligned}\tag{4.9}$$

Combining (4.4) and (4.5) (by (4.9) in case $b=0$), it follows that

$$||tA^{\gamma,b}P_t^{\gamma,b}f||_\infty \leq \frac{2(1+\sqrt{2}+b)}{\gamma}||f||_\infty \leq \frac{2(1+\sqrt{2}+B)}{\gamma_0}||f||_\infty,$$

for every $t\geq 0$, $b\in[0,B], \gamma\in[\gamma_0,\infty[$ and $f\in C([0,\infty])$. □

As a consequence, we obtain (see, e.g., [12, Chap. II, Theorem 4.6]).

Corollary 4.2. *Let $B,\gamma_0>0$. Then there exists $\theta_1=\theta(B,\gamma_0)\in]0,\frac{\pi}{2}]$ such that, for every $\theta\in]0,\theta_1[$ there exists $d_1=d(B,\gamma_0,\theta)>0$ for which*

$$||R(\lambda,A^{\gamma,b})f||_\infty \leq d_1\frac{||f||_\infty}{|\lambda|},\quad |\arg\lambda|\leq\frac{\pi}{2}+\theta,\, \lambda\neq 0, \tag{4.10}$$

for every $f\in C([0,\infty])$, $b\in[0,B]$ and $\gamma\geq\gamma_0$.

5. Gradient estimates

Proposition 5.1. *Let $B, \gamma_0 > 0$. Then, for every $b \in [0, B]$ and $\gamma \geq \gamma_0$, the following properties hold.*

(1) *For every $f \in C([0,\infty])$ and $t > 0$, $P_t^{\gamma,b} f \in C^1([0,\infty[)$ and*

$$|\sqrt{x}(P_t^{\gamma,b} f)'(x)| \leq \frac{2\sqrt{2}}{\sqrt{\gamma_0 t}} ||f||_\infty, \quad x \geq 0. \tag{5.1}$$

(2) *There exists $\theta_1 = \theta(B, \gamma_0) \in]0, \frac{\pi}{2}]$ such that, for every $\theta \in]0, \theta_1[$ there exists $d_2 = d(B, \gamma_0, \theta) > 0$ for which*

$$R(\lambda, A^{\gamma,b}) f \in C^1(]0,\infty[), \tag{5.2}$$

$$\lim_{x \to 0^+} \sqrt{x}(R(\lambda, A^{\gamma,b}) f)'(x) = 0, \tag{5.3}$$

$$|\sqrt{x}(R(\lambda, A^{\gamma,b}) f)'(x)| \leq d_2 \frac{||f||_\infty}{\sqrt{|\lambda|}}, \quad x > 0, \tag{5.4}$$

for every $f \in C([0,\infty])$ and $|\arg\lambda| \leq \frac{\pi}{2} + \theta$ with $\lambda \neq 0$, where d_1 is the constant appearing in (4.10).

Proof. (1) Let $b > 0$ and $\gamma > 0$. From (4.2) it follows immediately that $P_t^{\gamma,b} f \in C^1([0,\infty[)$. On the other hand, we have

$$\begin{aligned} \frac{1}{\Gamma(\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{\frac{b}{\gamma}-1} \left| z - \frac{b}{\gamma} \right| dz &\leq \frac{1}{\Gamma(\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{\frac{b}{\gamma}} dz + \frac{\frac{b}{\gamma}}{\Gamma(\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{\frac{b}{\gamma}-1} dz \\ &= 1 + \frac{\frac{b}{\gamma}}{\Gamma(\frac{b}{\gamma}+1)} \Gamma(\frac{b}{\gamma}) = 2. \end{aligned}$$

So, fixed any $f \in C([0,\infty])$ and applying Hölder inequality in (4.4) and Lemma 1.3, we get, for every $t > 0$ and $x > 0$, that

$$\begin{aligned} |(P_t^{\gamma,b} f)'(x)| &\leq ||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}-1} \left| z - m - \frac{b}{\gamma} \right| dz \\ &\leq ||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{\gamma t}} \left(2 + \sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\sqrt{m+\frac{b}{\gamma}}} \right) \\ &\leq ||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{\gamma t}} \left(2 + 2\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\sqrt{m+1}} \right) \\ &= 2||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{\gamma t}} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\sqrt{m+1}} \\ &\leq 2||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{\gamma t}} \left(\sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!(m+1)} \right)^{\frac{1}{2}} e^{\frac{x}{2\gamma t}} \\ &= 2||f||_\infty \frac{1}{\gamma t} e^{-\frac{x}{2\gamma t}} \left[\frac{\gamma t}{x} (e^{\frac{x}{\gamma t}} - 1) \right]^{\frac{1}{2}} \\ &= 2||f||_\infty \frac{1}{\sqrt{\gamma t x}} \left(1 - e^{-\frac{x}{2\gamma t}} \right)^{\frac{1}{2}} \leq 2||f||_\infty \frac{1}{\sqrt{\gamma t x}}. \end{aligned}$$

Now, let $b=0$. Then by (4.7) we have, for every $t>0$ and $x>0$, that

$$\begin{aligned}|(P_t^{\gamma,0}f)'(x)| &\leq 2||f||_\infty \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\left(1+\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\sqrt{m}}\right)\\ &\leq 2||f||_\infty \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\left(\sqrt{2}+\sum_{m=1}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{\sqrt{2}}{m!\sqrt{m+1}}\right)\\ &= 2\sqrt{2}||f||_\infty \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\sqrt{m+1}} \leq 2\sqrt{2}||f||_\infty \frac{1}{\sqrt{\gamma t x}}.\end{aligned}$$

So, (1) is satisfied.

(2) By (1) and the dominated convergence theorem we obtain, for every $\eta>1$ and $x>0$, that

$$\sqrt{x}(R(\eta,A^{\gamma,b})f)'(x)=\sqrt{x}D\left(\int_0^{+\infty} e^{-\eta t}P_t^{\gamma,b}f\,dt\right)=\int_0^\infty e^{-\eta t}\sqrt{x}(P_t^{\gamma,b}f)'dt.$$

At this point, (5.3) and (5.4) follow by arguing as in the proof of [1, Proposition 2.1]. $\square$

In case $b>0$, we can achieve the following global gradient estimate for the resolvent.

Proposition 5.2. *Let $b,\gamma>0$. Then there exists a constant $C>0$ independent on b such that, for every $\lambda>0$ and $f\in C([0,\infty])$, we have*

$$||(R(\lambda,A_b)f)'||_\infty \leq \frac{1}{\gamma}\max\left\{2,\frac{\gamma}{b}\right\}C||f||_\infty.$$

In order to prove this, we need the following result.

Lemma 5.3. *Let $f\in C^1([0,\infty[)\cap C([0,\infty])$, with f' bounded. Then, for every $x\geq 0$, we have*

$$(P_t^{\gamma,b}f)'(x)=e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)}\int_0^\infty f'(\gamma zt)z^{m+\frac{b}{\gamma}}e^{-z}dz.$$

Proof. Integrating by parts in (4.2), we obtain, for every $x\geq 0$ and $t>0$, that

$$\begin{aligned}(P_t^{\gamma,b}f)'(x) &= \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!}\int_0^\infty e^{-z}f(\gamma zt)\left(\frac{z^{m+b/\gamma}}{\Gamma(m+b/\gamma+1)}-\frac{z^{m+b/\gamma-1}}{\Gamma(m+b/\gamma)}\right)dz\\ &= \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!}\left(\int_0^\infty e^{-z}f(\gamma zt)\frac{z^{m+b/\gamma}}{\Gamma(m+b/\gamma+1)}dz-\int_0^\infty e^{-z}f(\gamma zt)\frac{z^{m+b/\gamma-1}}{\Gamma(m+b/\gamma)}dz\right)\\ &= \frac{e^{-\frac{x}{\gamma t}}}{\gamma t}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!}\left(\left[-e^{-z}f(\gamma zt)\frac{z^{m+b/\gamma}}{\Gamma(m+b/\gamma+1)}\right]_0^\infty+\int_0^\infty \gamma te^{-z}f'(\gamma zt)\frac{z^{m+b/\gamma}}{\Gamma(m+b/\gamma+1)}dz\right.\\ &\quad\left.+\int_0^\infty e^{-z}f(\gamma zt)(m+b/\gamma)\frac{z^{m+b/\gamma-1}}{\Gamma(m+b/\gamma+1)}dz-\int_0^\infty e^{-z}f(\gamma zt)\frac{z^{m+b/\gamma-1}}{\Gamma(m+b/\gamma)}dz\right)\\ &= e^{-\frac{x}{\gamma t}}\sum_{m=0}^\infty\left(\frac{x}{\gamma t}\right)^m\frac{1}{m!\Gamma(m+b/\gamma+1)}\int_0^\infty f'(\gamma zt)z^{m+b/\gamma}e^{-z}dz.\qquad\square\end{aligned}$$

We observe that the above inequality ensures that, for every $f\in C^1([0,\infty[)\cap C([0,\infty])$, with f' bounded, we have

$$||(R(\lambda,A^{\gamma,b})f)'||_\infty\leq||f'||_\infty,\quad \lambda>0.$$

But in order to estimate $||(R(\lambda,A^{\gamma,b})f)'||_\infty$ with $||f||_\infty$, we need to proceed as follows.

Proof of Proposition 5.2. We first assume that $f \in C^1([0,\infty[) \cap C([0,\infty])$, with f' bounded. In such a case, by Lemma 5.3 we have, for every $x \geq 0$ and $\lambda > 0$, that

$$
\begin{aligned}
&(R(\lambda, A^{\gamma,b})f)'(x) = \int_0^\infty e^{-\lambda t}(P_t^{\gamma,b}f)'(x)dt \\
&= \int_0^\infty e^{-\lambda t} e^{-\frac{x}{\gamma t}} \left(\sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!} \frac{1}{\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty f'(\gamma z t) z^{m+\frac{b}{\gamma}} e^{-z} dz \right) dt \\
&= \sum_{m=0}^\infty \int_0^\infty dz \left(e^{-z} z^{m+\frac{b}{\gamma}} \frac{1}{m!} \frac{1}{\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m f'(\gamma z t) e^{-\frac{x}{\gamma t}} dt \right).
\end{aligned}
$$

On the other hand, for every $m \geq 0$, $x \geq 0$ and $\lambda > 0$, we have

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m f'(\gamma z t) e^{-\frac{x}{\gamma t}} dt = \\
&= \left[\frac{f(\gamma z t)}{\gamma z} e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m e^{-\frac{x}{\gamma t}} \right]_0^\infty - \frac{1}{\gamma z} \int_0^\infty f(\gamma z t) \left[e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m e^{-\frac{x}{\gamma t}} \right]' dt \\
&= \frac{1}{\gamma z} \int_0^\infty e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m e^{-\frac{x}{\gamma t}} \left(\lambda + \frac{m}{t} - \frac{x}{\gamma t^2} \right) f(z\gamma t) dt.
\end{aligned}
$$

So, we obtain, for every $x \geq 0$ and $\lambda > 0$, that

$$
\begin{aligned}
&(R(\lambda, A^{\gamma,b})f)'(x) = \\
&= \frac{1}{\gamma} \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)} \int_0^\infty dz e^{-z} z^{m+\frac{b}{\gamma}-1} \int_0^\infty f(\gamma z t) e^{-\lambda t} \left(\frac{x}{\gamma t}\right)^m \times \\
&\times e^{-\frac{x}{\gamma t}} \left(\lambda + \frac{m}{t} - \frac{x}{\gamma t^2} \right) dt \\
&= \frac{1}{\gamma} \int_0^\infty dt e^{-\lambda t} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!\Gamma(m+\frac{b}{\gamma}+1)} e^{-\frac{x}{\gamma t}} \left(\lambda + \frac{m}{t} - \frac{x}{\gamma t^2} \right) \times \\
&\times \int_0^\infty e^{-z} z^{m+\frac{b}{\gamma}-1} f(\gamma z t) dz,
\end{aligned}
$$

after having observed that we can interchange sums and integrals because the integrals and series are absolutely summable.

By Lemma 1.2 it follows, for every $x \geq 0$ and $\lambda > 0$, that

$$
\begin{aligned}
&|R(\lambda, A^{\gamma,b})f)'(x)| \leq \frac{||f||_\infty}{\gamma} \int_0^\infty dt e^{-\lambda t} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{m!(m+\frac{b}{\gamma})} e^{-\frac{x}{\gamma t}} \left| \lambda + \frac{m}{t} - \frac{x}{\gamma t^2} \right| \\
\leq\ & \frac{||f||_\infty}{\gamma} \left(\frac{\gamma}{b} \int_0^\infty \lambda e^{-\lambda t} dt + \max\{2, \frac{\gamma}{b}\} \int_0^\infty \frac{e^{-\lambda t}}{t} \sum_{m=0}^\infty \left(\frac{x}{\gamma t}\right)^m \frac{1}{(m+1)!} e^{-\frac{x}{\gamma t}} \left| m - \frac{x}{\gamma t} \right| dt \right) \\
\leq\ & \frac{1}{\gamma} \max\left\{2, \frac{\gamma}{b}\right\} ||f||_\infty \left(1 + \int_0^\infty e^{-\lambda \frac{x}{\gamma s}} \frac{1}{s} \sum_{m=0}^\infty s^m e^{-s} |m-s| \frac{1}{(m+1)!} ds \right) \\
\leq\ & \frac{1}{\gamma} \max\left\{2, \frac{\gamma}{b}\right\} C ||f||_\infty
\end{aligned}
$$

with C independent on b.

Finally, let $f \in C([0,\infty])$ and let $(f_n)_n \subseteq C^1([0,\infty])$ be an approximating sequence for f in $C([0,\infty])$. Then from (5.4) it follows that

$$\lim_{n\to\infty} \sqrt{x}\partial_x R(\lambda, A^{\gamma,b}) f_n(x) = \sqrt{x}\partial_x R(\lambda, A^{\gamma,b}) f(x)$$

for every $x \geq 0$ and $\lambda > 0$. So, for every $x > 0$ and $\lambda > 0$, we obtain

$$|(R(\lambda, A^{\gamma,b})f)'(x)| = \lim_{n\to\infty} |(R(\lambda, A^{\gamma,b})f_n)'(x)|$$
$$\leq \lim_{n\to\infty} \frac{1}{\gamma} \max\left\{2, \frac{\gamma}{b}\right\} ||f_n||_\infty = \frac{1}{\gamma} \max\left\{2, \frac{\gamma}{b}\right\} ||f||_\infty. \qquad \square$$

Corollary 5.4. *Let $b, \gamma > 0$. Then there exists a constant $C > 0$ independent on b such that, for every $\lambda > 0$ and $f \in C([0,\infty])$, we have*

$$||x(R(\lambda, A_b)f)''||_\infty \leq \frac{C}{\gamma}\left[1 + \max\left\{2\frac{b}{\gamma}, 1\right\}\right] ||f||_\infty.$$

Proof. Combining Corollary 4.2 and Proposition 5.2 we obtain, for every $\lambda > 0$ and $f \in C([0,\infty])$, that

$$\begin{aligned}
|x(R(\lambda, A^{\gamma,b})f)''(x)| &= \left|\frac{1}{\gamma}\left[A^{\gamma,b}(R(\lambda, A^{\gamma,b}f) - b(R(\lambda, A^{\gamma,b})f)'(x)\right]\right| \\
&= \left|\frac{1}{\gamma}\left[\lambda R(\lambda, A^{\gamma,b})f - f - b(R(\lambda, A^{\gamma,b})f)'(x)\right]\right| \\
&\leq \frac{1}{\gamma}\left[1 + d_1 + \frac{b}{\gamma}\max\left\{2, \frac{\gamma}{b}\right\} C\right] ||f||_\infty \\
&\leq \frac{C'}{\gamma}\left[1 + \max\left\{2\frac{b}{\gamma}, 1\right\}\right] ||f||_\infty,
\end{aligned}$$

where $C' := \max\{1 + d_1, C\}$. $\square$

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